

Coarsening with a Frozen Vertex

Michael Damron^{1,*}, Hana Kogan²,

Charles M. Newman^{3,4,†}, Vidas Sidoravicius^{3,4}

¹ Georgia Institute of Technology, Atlanta GA and Indiana University,
Bloomington IN

² CUNY College of Staten Island

³ Courant Institute of Mathematical Sciences

⁴ NYU–ECNU Institute of Mathematical Sciences at NYU Shanghai

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Abstract

In the standard nearest-neighbor coarsening model with state space $\{-1, +1\}^{\mathbb{Z}^2}$ and initial state chosen from symmetric product measure, it is known (see [2]) that almost surely, every vertex flips infinitely often. In this paper, we study the modified model in which a single vertex is frozen to $+1$ for all time, and show that every other site still flips infinitely often. The proof combines stochastic domination (attractivity) and influence propagation arguments.

1 Introduction

As in our earlier paper [1], we study and compare the long time behavior of two continuous time Markov coarsening models with state space $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$. One, $\sigma(t)$, is the standard model in which at time zero $\{\sigma_x(0) : x \in \mathbb{Z}^d\}$ is an i.i.d. set with $\theta \equiv P(\sigma_x(0) = +1) = 1/2$ and then vertices update to agree with a strict majority of their $2d$ nearest neighbors or, in case of a tie, choose their value by tossing a fair coin. The modified model, $\sigma'(t)$, is the same except that σ' at the origin $(0, 0, \dots, 0)$ is frozen to $+1$ for all $t \geq 0$.

For $d = 2$, it is an old result [2] that in the standard $\sigma(t)$ model, almost surely, every vertex changes sign infinitely many times as $t \rightarrow \infty$. The main result of this paper (see Theorem 2.7) is that the same is true for the frozen model $\sigma'(t)$ on \mathbb{Z}^2 . It is believed (see, for example, Sec. 6.2 of [3]), but not proved, that the $d = 2$ behavior of σ remains valid at least for some values of $d > 2$. If this were so, then the arguments of this paper would show the same for the corresponding σ' model.

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In the previous paper [1] we considered models with infinitely many frozen vertices and in this paper a model with a single frozen vertex. It would be of interest to study models with finitely many, but more than one, frozen vertices; in this regard, see the remark following the proof of Theorem 2.8 below.

2 Results

In this section we fix $d = 2$. We also use the standard convention that the updates are made when independent rate one Poisson process clocks at each vertex ring.

Let A_T denote the event that the “right” neighbor of the origin (at $x = (1, 0)$) is -1 for some $t \geq T$. Let $A_T^1 \subset A_T$ denote the event that the right neighbor of the origin is the *first* neighbor to be -1 at some time $t \geq T$ (more precisely, that no other neighbor is -1 at an earlier time in $[T, \infty]$). Let $B_{L,s}$ for $s \in \{-1, +1\}^{\Lambda_L}$ (where $\Lambda_L = \{-L, -L+1, \dots, L\}^2$) denote the event that $\sigma'(0)|_{\Lambda_L} = s$ and write $B_{L,+}$ when $s \equiv +1$. We denote the probability measure for the frozen origin $\sigma'(\cdot)$ model by P' and that for the regular coarsening model $\sigma(\cdot)$ by P .

Lemma 2.1. *For all L ,*

$$P(A_0^1 | B_{L,+}) \geq 1/4.$$

Proof. The result is an easy consequence of symmetry among the four neighbors of the origin and the fact that $P(A_0) = 1$ (indeed, for all T , $P(A_T) = 1$ — see [2]). \square

Let Σ_T^L denote the sigma-field generated by the initial spin values and clock rings and coin tosses up to time T inside the box Λ_L .

Proposition 2.2. *For any T, L ,*

$$P'(A_T | \Sigma_T^L) \geq 1/4 \text{ a.s.}$$

Proof. Let $\tilde{\sigma}_T^L(\cdot)$ denote the model with the spin values at all sites in Λ_L frozen to $+1$ from time 0 up to time T and with the spin value at the origin remaining frozen at $+1$ thereafter. Denote the corresponding probability measure by \tilde{P}_T^L . Under the standard coupling, $\tilde{\sigma}(\cdot)$ stochastically dominates $\sigma'(\cdot)$, so we have

$$P'(A_T | \Sigma_T^L) \geq \tilde{P}_T^L(A_T) \geq \tilde{P}_T^L(A_T^1).$$

To continue the proof, we will use the following result about the “propagation speed” of influence between different spatial regions:

Lemma 2.3. *Let D_T^L denote the event that $\sigma_x(t) = +1 \forall x \in \Lambda_L, \forall t \in [0, T]$. Then*

$$\forall L, T, \varepsilon, \exists L' \text{ such that } P(D_T^L | B_{L',+}) \geq 1 - \varepsilon.$$

Proof. Let $L' \gg L$ and note that given $B_{L',+}$, $(D_T^L)^c$ can occur only if there is a nearest neighbor (self-avoiding) path between the boundaries of the two sets, $\mathbb{Z}^2 \setminus \Lambda_{L'}$ and Λ_L , along which there are clock rings occurring in succession between times 0 and T . Any such path is at least of length $L' - L$ (i.e., contains at least $L' - L$ vertices besides the starting one).

Consider a particular path γ of length $m \geq L' - L$. For each m there are no more than 3^m such paths from each boundary point and the time it takes for successive clock rings along γ is at

least $S_m = \sum_{i=1}^m \tau_i$ where the τ_i are i.i.d. exponential random variables with parameter 1. By the exponential Markov inequality, for any $\alpha > 0$,

$$P\left(\sum_{i=1}^m \tau_i < T\right) = P\left(-\sum_{i=1}^m \tau_i > -T\right) \leq \frac{E(e^{-\alpha \sum_{i=1}^m \tau_i})}{E(e^{-\alpha T})} = e^{\alpha T} E\{e^{-\alpha \tau_i}\}^m = \frac{e^{\alpha T}}{(1 + \alpha)^m}.$$

Therefore, since there are at most CL' possible starting points (for some constant C),

$$P((D_T^L)^c | B_{L',+}) \leq CL' \sum_{m=L'-L}^{\infty} 3^m \frac{e^{\alpha T}}{(1 + \alpha)^m} = C(\alpha, T, L) L' \left(\frac{3}{1 + \alpha}\right)^{L'},$$

where $C(\alpha, T, L)$ is a constant depending on α , T and L . Taking $\alpha > 2$ and the limit as $L' \rightarrow \infty$ completes the proof of the lemma.

Proof. We continue the proof of Proposition 2.2. Pick $\epsilon > 0$ and fix T and L . By Lemma 2.3, $\exists L'$ such that

$$P(D_T^L | B_{L',+}) \geq 1 - \epsilon.$$

Therefore, given $B_{L',+}$, with probability at least $1 - \epsilon$, $\sigma_t(\cdot)$ positively dominates $\tilde{\sigma}_L^T(\cdot)$ for $0 \leq t < S$, where $S = \inf\{t > 0 \mid \sigma_t(0, 0) = -1\}$, and so

$$\tilde{P}_L^T(A_T^1) \geq P(A_T^1 | B_{L',+}) - \epsilon \geq 1/4 - \epsilon.$$

Taking the limit as $\epsilon \rightarrow 0$ completes the proof of Proposition 2.2. \square

Now let Σ_T denote the sigma field generated by the initial assignment of spins on \mathbb{Z}^2 and the clock rings and coin tosses on \mathbb{Z}^2 up to time T .

Proposition 2.4. *For all T ,*

$$P'(A_T | \Sigma_T) \geq 1/4 \text{ a.s.}$$

Proof. For $L \geq 1$ let $X_L = P'(A_T | \Sigma_T^L)$. $\{\Sigma_T^L, L \geq 1\}$ is an increasing filtration of sigma fields, and $E(X_{L+1} | \Sigma_T^L) = X_L$. By the martingale convergence theorem, $\lim_{L \rightarrow \infty} (X_L) = X_\infty = P'(A_T | \Sigma_T)$ and since $X_L \geq 1/4$ for all L , we have $P'(A_T | \Sigma_T) \geq 1/4$. \square

Let $A_{T,T'}$ denote the event that the right neighbor of the origin is -1 for some time $t \in [T, T']$. The following is immediate from Proposition 2.4.

Corollary 2.5.

$$\lim_{T' \rightarrow \infty} P'(A_{T,T'} | \Sigma_T) \geq 1/4 \text{ a.s.}$$

Lemma 2.6. *For any $T \geq 0$ and $\gamma > 0$, \exists a deterministic T' such that*

$$P\{\omega : P'(A_{T,T'} | \Sigma_T) \geq 1/8\} \geq 1 - \gamma.$$

Proof. This is a straightforward consequence of the preceding corollary. \square

Theorem 2.7. *For any T ,*

$$P'(A_T) = 1, \text{ and hence } P'(\cap_{T>0} A_T) = 1.$$

It follows that with probability one, $\sigma'_{(1,0)}(t)$ changes sign infinitely many times as $t \rightarrow \infty$.

Proof. Given T and $\epsilon > 0$ construct a sequence of deterministic times $\{T_i; i \geq 0\}$ so that

1. $T_0 = T$, and
2. $P'\{\omega : P'(A_{T_{i-1}, T_i} | \Sigma_{T_{i-1}}) \geq 1/8\} \geq 1 - \frac{\epsilon}{2^i}$.

Condition now on the event (of probability at least $1 - \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = 1 - \epsilon$) that $P'(A_{T_{i-1}, T_i} | \Sigma_{T_{i-1}}) \geq 1/8$ for all i . On this conditioned probability space, letting $\tilde{W}_i = 1$ (and otherwise 0) if A_{T_{i-1}, T_i} occurs, we note that the \tilde{W}_i 's stochastically dominate i.i.d. $\{0, 1\}$ -valued W_i 's with $\text{Prob}(W_i = 1) = 1/8$. Thus

$$P'(A_{T_{i-1}, T_i} \text{ occurs for only finitely many } i) \leq \epsilon.$$

Letting $\epsilon \rightarrow 0$ completes the proof of the first part of the theorem. The second part then follows because by stochastic domination (attractivity) and the results of [2], $\sigma'_{(0,0)}(t_i)$ equals $+1$ for an infinite sequence of $t_i \rightarrow \infty$. \square

The next theorem follows from a modified version of the proof of Theorem 2.7.

Theorem 2.8. *Every site in $\{\mathbb{Z}^2 \setminus (0, 0)\}$ flips infinitely many times in $\sigma'(\cdot)$ with probability one.*

Proof. For any site z other than the origin, and for L much larger than say the Euclidean norm of z , we consider the unfrozen σ model in which at time zero all the vertex values are set to $+1$ in the box of side length $2L$, centered at $z/2$ (so that the origin and z are located symmetrically with respect to this box). Then with probability $1/2$ the vertex at z flips to -1 before the one at the origin flips and until just after that time, there is no difference between the frozen (at the origin) σ' model and the unfrozen σ model. Hence there is probability at least $1/2$ in σ' that z will flip to minus. By applying the methods used in the proof of Theorem 2.7 (but with $1/4$ now replaced by $1/2$), we conclude that z will flip infinitely many times with probability one. \square

We note that the line of reasoning in the proof of the last theorem could have also been used to give a modified proof of Theorem 2.7 with $1/4$ replaced by $1/2$. A more interesting remark is the following.

Remark 2.9. *For the process σ'' with some finite set \mathcal{S} of vertices frozen to $+1$, it is possible to show by an extension of the arguments used in this paper that there is a finite deterministic $\mathcal{S}' \supseteq \mathcal{S}$ such that all sites in $\{\mathbb{Z}^2 \setminus \mathcal{S}'\}$ flip infinitely many times in $\sigma''(\cdot)$ with probability one. In some cases, \mathcal{S}' must be strictly larger than \mathcal{S} — e.g., when $\mathcal{S} = \{(-L, -L), (-L, L), (+L, -L), (+L, L)\}$, \mathcal{S}' includes all of Λ_L . One may also consider processes where some vertices are frozen to -1 and some to $+1$. We expect to pursue these issues in a future paper.*

References

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